

NONIMMERSIONS AND NONEMBEDDINGS OF QUATERNIONIC SPHERICAL SPACE FORMS

BY

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ABSTRACT. We determine the orders of the canonical elements in KO -rings of quaternionic spherical space forms S^{4n+3}/Q_k and apply them to prove the nonexistence theorems of immersions and embeddings of S^{4n+3}/Q_k in Euclidean spaces.

1. Statements of results. Let $Q_k = [x, y: x^{2^{k-2}} = y^2, xyx = y]$ be the generalized quaternion group of order 2^k ($k > 2$). (Note that the relation $x^{2^{k-1}} = 1$ follows from the above two relations.) Let $d_1: Q_k \rightarrow S^3 = \text{Sp}(1) = \text{SU}(2)$ be the natural inclusion defined by $d_1(x) = \exp(2\pi i/2^{k-1})$, $d_1(y) = j$. Then Q_k acts freely on the unit sphere S^{4n+3} in the quaternion $(n+1)$ -space H^{n+1} by the diagonal action $(n+1)d_1: Q_k \rightarrow \text{Sp}(n+1)$. The quotient manifold S^{4n+3}/Q_k is called the quaternionic spherical space form. D. Pitt [8] studied the structure of K - and KO -rings of S^{4n+3}/Q_k and considered the problem of immersing or embedding S^{4n+3}/Q_k in Euclidean space R^m using the techniques of M. F. Atiyah [1] (cf. also [5, Chapter 6] and [6, Chapter 3]).

The purpose of this note is to determine the orders of the canonical elements in $\widetilde{KO}(S^{4n+3}/Q_k)$ and apply them to improve the nonexistence theorems of immersions and embeddings of S^{4n+3}/Q_k . Let $M \not\subseteq R^m$ (or $M \not\subset R^m$) denote nonexistence of a C^∞ -immersion (or a C^∞ -embedding) of M in R^m . Let $\nu(n)$ be the nonnegative integer such that $n = q \cdot 2^{\nu(n)}$, where q is odd. Our main theorem is

THEOREM 1.1. *If $\nu(2^{n+1+i}) < 2n + k - 2i + \epsilon$, then $S^{4n+3}/Q_k \not\subseteq R^{4n+2+2i}$ and $S^{4n+3}/Q_k \not\subset R^{4n+3+2i}$, where $\epsilon = 0$ if n is even > 0 , and $\epsilon = 1$ if n is odd.*

Define

$$N(n, k) = \max \left[i: 1 \leq i \leq n, \nu(2^{n+1+i}) < 2n + k - 2i + \epsilon \right].$$

The case $N(n, k) = n$ was obtained by Pitt [8, Corollary 5.6], and the case $k = 3$ was obtained by K. Fujii. It follows from Theorem 1.1, for example, that

$$\begin{aligned} S^{15}/Q_k \not\subseteq R^{20}, \quad S^{15}/Q_k \not\subset R^{21}; \quad S^{31}/Q_3 \not\subseteq R^{42}, \quad S^{31}/Q_3 \not\subset R^{43}, \\ S^{31}/Q_k \not\subseteq R^{44}, \quad S^{31}/Q_k \not\subset R^{45} \quad \text{for } k \geq 4. \end{aligned}$$

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The complex representation ring $R_C(Q_k)$ of Q_k is generated as a free abelian group by $1, a, b, c$ and d_r ($r = 1, 2, \dots, 2^{k-2} - 1$) defined below (cf. [2, §47.15; 8, §1 and 3, §3]):

$$\begin{cases} 1(x) = 1, & \begin{cases} a(x) = 1, & \begin{cases} b(x) = -1, & \begin{cases} c(x) = -1, \\ c(y) = -1, \end{cases} \end{cases} \\ 1(y) = 1, & \begin{cases} a(y) = -1, & \begin{cases} b(y) = 1, \end{cases} \end{cases} \end{cases}$$

$$d_r(x) = \begin{bmatrix} \omega^r & 0 \\ 0 & \omega^{-r} \end{bmatrix}, \quad d_r(y) = \begin{bmatrix} 0 & (-1)^r \\ 1 & 0 \end{bmatrix},$$

where ω is a primitive 2^{k-1} th root $\exp(2\pi i/2^{k-1})$ of unity. The multiplicative structure is given by

$$a^2 = b^2 = c^2 = 1, \quad ab = c, \quad d_r d_s = d_{r+s} + d_{r-s}, \quad bd_r = d_{2^{k-2}-r},$$

where we define

$$d_0 = 1 + a, \quad d_{2^{k-2}} = b + c, \quad d_{-r} = d_r, \quad d_{2^{k-2}+r} = d_{2^{k-2}-r}.$$

The reduced representation ring $\tilde{R}_C(Q_k)$ is generated as a free abelian group by $\alpha = a - 1, \beta = b - 1, \gamma = a + b + c - 3, \delta_r = d_r - 2$ ($r = 1, 2, \dots, 2^{k-2} - 1$) with relations (cf. [3, Proposition 3.3]):

$$\begin{aligned} \alpha^2 &= -2\alpha, \quad \beta^2 = -2\beta, \quad \gamma = \alpha\beta + 2\alpha + 2\beta, \quad \alpha\delta_1 = -2\alpha, \\ \beta\delta_1 &= -2\beta + \delta_{2^{k-2}-1} - \delta_1, \quad \delta_{r+1} = \delta_1\delta_r + 2\delta_1 + 2\delta_r - \delta_{r-1} \end{aligned}$$

where $\delta_{2^{k-2}} = \gamma - \alpha, \delta_0 = \alpha$. Thus $\tilde{R}_C(Q_k)$ is generated by α, β and δ_1 as a ring.

Let $c_R: R_R(Q_k) \rightarrow R_C(Q_k)$ be the complexification. The real representation ring $R_R(Q_k)$, considered as the subring $c_R(R_R(Q_k))$ of $R_C(Q_k)$, is generated by $1, a, b, c, d_{2r}$ and $2d_{2r+1}$ ($r = 0, 1, \dots, 2^{k-3} - 1$) (cf. [8, Proposition 1.5]).

Define elements v and z in $\tilde{R}_R(Q_k)$ by

$$(1.2) \quad c_R^{-1}(2\delta_1) = v, \quad c_R^{-1}(\delta_1^2) = z.$$

Let λ be the canonical complex plane bundle over the quaternion projective space $HP^n = S^{4n+3}/S^3$, and let $\pi: S^{4n+3}/Q_k \rightarrow HP^n$ be the natural projection. Let $\xi_C: \tilde{R}_C(Q_k) \rightarrow \tilde{K}(S^{4n+3}/Q_k)$ be the projection defined in [3, §4] and put $\delta = \xi_C(\delta_1)$. Then we have $\delta = \pi^*\lambda - 2$ (cf. [3, Lemma 4.4]). The order $\#\delta^i$ of $\delta^i \in \tilde{K}(S^{4n+3}/Q_k)$ is determined by H. Ōshima in [7, Proposition 5.2] and T. Mormann in [6, Chapter 2, Theorem 4.52] as follows.

PROPOSITION 1.3. $\#\delta^i = 2^{2n+k-2i}$ ($1 \leq i \leq n$).

Let $r_C: \tilde{K}(X) \rightarrow \widetilde{KO}(X)$ and $c_R: \widetilde{KO}(X) \rightarrow \tilde{K}(X)$ be the realification and the complexification, respectively. Let $\xi_R: \tilde{R}_R(Q_k) \rightarrow \widetilde{KO}(S^{4n+3}/Q_k)$ be the projection defined in [4, (3.9)] (or in [8, Theorem 2.5]). Then, by (1.2),

$$\xi_R v = r_C(\pi^*\lambda - 2) \quad \text{and} \quad \xi_R z = c_R^{-1}((\pi^*\lambda - 2)^2)$$

(cf. [4, Lemma 3.10]), because δ_1 is self-conjugate and $c_R r_C = 1 + \text{conjugation}$. For simplicity we write v and z instead of $\xi_R v$ and $\xi_R z$. Then, for the complexification $c_R: \widetilde{KO}(S^{4n+3}/Q_k) \rightarrow \tilde{K}(S^{4n+3}/Q_k)$, we have

$$(1.4) \quad c_R(v) = 2\delta, \quad c_R(z) = \delta^2.$$

Let $\#a$ (or $\#A$) denote the order of an element a (or a group A). The orders of the canonical elements in $\widetilde{KO}(S^{4n+3}/Q_k)$ are determined as follows.

THEOREM 1.5. For z^i and $vz^i \in \widetilde{KO}(S^{8m+7}/Q_k)$,

$$\#z^i = 2^{4m+k-4i+3} \quad (0 < i \leq m),$$

$$\#vz^i = 2^{4m+k-4i} \quad (0 \leq i \leq m).$$

THEOREM 1.6. For z^i and $vz^i \in \widetilde{KO}(S^{8m+3}/Q_k)$,

$$\#z^i = 2^{4m+k-4i} \quad (0 < i \leq m),$$

$$\#vz^i = 2^{4m+k-4i-3} \quad (0 \leq i < m).$$

COROLLARY 1.7. For $v \in \widetilde{KO}(S^{4n+3}/Q_k)$,

$$\#v = \begin{cases} 2^{2n+k-2} & \text{if } n \text{ is odd,} \\ 2^{2n+k-3} & \text{if } n \text{ is even } > 0. \end{cases}$$

K. Fujii [4] proved the result for $k = 3$. H. Ōshima [7] announced it for $k = 4$ and conjectured Corollary 1.7.

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2. Proofs of Theorems 1.5 and 1.6. First we prepare a lemma.

LEMMA 2.1. In $\widetilde{KO}(S^{4n+3}/Q_k)$:

$$(a) \quad 2^k v + \sum_{j=1}^{2^{k-3}} (2c_{2j} + c_{2j+1}v)z^j = 0,$$

$$(b) \quad 2^k z + \sum_{j=1}^{2^{k-3}} (2^{-1}c_{2j}v + c_{2j+1}z)z^j = 0,$$

where c_s are integers satisfying

$$(*) \quad \begin{aligned} c_2 &= 2^{k-2}(2^{2^{k-3}} + 1)/3, & c_{2^{k-2}+1} &= 1, \\ \nu(c_s) &\geq \max(1, k-s) & \text{for } 0 < s \leq 2^{k-2}. \end{aligned}$$

PROOF. It was proved in [6] that $\sum_{s=1}^{2^{k-2}+1} c_s \delta_1^s = 0$ in $\tilde{R}_C(Q_k)$, where

$$c_s = \binom{2^{k-2}+s}{2s-1} + 2 \binom{2^{k-2}+s-1}{2s-1} + \binom{2^{k-2}+s-2}{2s-1}.$$

(*) follows easily from

$$\nu \binom{m}{n} = \alpha(n) + \alpha(m-n) - \alpha(m),$$

where $\alpha(n)$ denotes the number of nonzero terms in the dyadic expansion of n . (a) and (b) are proved by multiplying this by 2 and δ_1 , respectively, and applying $c_R^{-1}\xi_C$.

LEMMA 2.2. In $\widetilde{KO}(S^{8m+7}/Q_k)$:

- (i) $z^{m+1} = 0$,
- (ii) $2^{4i+k}vz^{m-i} = 0 \quad (0 \leq i \leq m)$,
- (iii) $2^{4i+k+3}z^{m-i} = 0 \quad (0 \leq i < m)$,
- (iv) $2^{4i+k-1}vz^{m-i} + 2^{4i+k+2}z^{m-i} = 0 \quad (0 \leq i < m)$,
- (v) $2^{4i+k+2}z^{m-i} + 2^{4i+k+3}vz^{m-i-1} = 0 \quad (0 \leq i < m)$.

PROOF. (i) follows from [8, Theorem 2.5].

(ii) and (iii) are proved by induction on i . (ii) for $i = 0$ follows from (i) and (a) $\times z^m$. (iii) for $i = 0$ follows from (b) $\times 2^3z^{m-1}$ and (ii) for $i = 0$. (iii) for any $j \leq i - 1$, (ii) for any $j \leq i$ and (b) $\times 2^{4i+3}z^{m-i-1}$ imply (iii) for $j = i$. (ii) for any $j \leq i$, (iii) for any $j \leq i$ and (a) $\times 2^{4i+4}z^{m-i-1}$ imply (ii) for $j = i + 1$.

Using (b) $\times 2^{4i+2}z^{m-i-1}$ (resp. (a) $\times 2^{4i+3}z^{m-i-1}$) and (i) $\sim$ (iii), we obtain (iv) (resp. (v)).

LEMMA 2.3. In $\widetilde{KO}(S^{8m+3}/Q_k)$:

- (i) $z^{m+1} = 0$,
- (ii) $vz^m = 0$,
- (iii) $2^{4i+k}z^{m-i} = 0 \quad (0 \leq i < m)$,
- (iv) $2^{4i+k+1}vz^{m-i-1} = 0 \quad (0 \leq i < m)$,
- (v) $2^{4i+k-1}z^{m-i} + 2^{4i+k}vz^{m-i-1} = 0 \quad (0 \leq i < m)$,
- (vi) $2^{4i+k}vz^{m-i-1} + 2^{4i+k+3}z^{m-i-1} = 0 \quad (0 \leq i < m - 1)$.

PROOF. (i) follows from Lemma 2.2(i) and the naturality. (ii) is proved in [4, §4]. The proofs of (iii) $\sim$ (vi) are similar to those of Lemma 2.2(ii) $\sim$ (v), so we omit the details.

PROOF OF THEOREM 1.5. By Lemma 2.2(iii), (ii) we have

$$\#z^i \leq 2^{4m+k-4i+3} \quad (0 < i \leq m) \quad \text{and} \quad \#vz^i \leq 2^{4m+k-4i} \quad (0 \leq i \leq m).$$

Let $j: S^{8m+3}/Q_k \rightarrow S^{8m+7}/Q_k$ be the natural inclusion. Then it follows from [4, §4] that $\text{Ker } j^*$, the kernel of the induced homomorphism $j^*: \widetilde{KO}(S^{8m+7}/Q_k) \rightarrow \widetilde{KO}(S^{8m+3}/Q_k)$, is generated by vz^m . According to [5, Chapter 6, Proposition 5.7],

$$\#\widetilde{KO}(S^{8m+7}/Q_k) = 2^{2mk+4m+k+4} \quad \text{and} \quad \#\widetilde{KO}(S^{8m+3}/Q_k) = 2^{2mk+4m+4}.$$

Hence, by Lemma 2.2(ii), we obtain

$$2^k = \#\widetilde{KO}(S^{8m+7}/Q_k) / \#\widetilde{KO}(S^{8m+3}/Q_k) \leq \# \text{Ker } j^* = \#vz^m \leq 2^k.$$

Thus $\#vz^m = 2^k$. Therefore, by Lemma 2.2(iv), (v) we have

$$2^{4m+k-4i+2}z^i = -2^{4m+k-4i-1}vz^i = \dots = -2^{k-i}vz^m \neq 0.$$

PROOF OF THEOREM 1.6. By Lemma 2.3(iii), (iv) we have

$$\#z^i \leq 2^{4m+k-4i} \quad (0 < i \leq m) \quad \text{and} \quad \#vz^i \leq 2^{4m+k-4i-3} \quad (0 \leq i < m).$$

By Lemma 2.3(vi), (v) we have

$$2^{4m+k-4i-1}z^i = -2^{4m+k-4i-4}vz^i = \dots = -2^{4m+k-4}v.$$

But, by (1.4) and Proposition 1.3, $c_R(2^{4m+k-4}v) = 2^{4m+k-3}\delta \neq 0$, so $2^{4m+k-4}v \neq 0$.

3. Atiyah's criterion. Exterior power operation $\lambda^i(\alpha)$, $\alpha \in R_C(Q_k)$, is determined by

$$\lambda^0(\alpha) = 1, \quad \lambda^1(\alpha) = \alpha, \quad \lambda^i(\alpha) = 0 \quad \text{for } i > 1, \alpha = 1, a, b, c,$$

$$\lambda^2(d_r) = 1 \quad \text{for } r \text{ odd}, \quad \lambda^2(d_r) = a \quad \text{for } r \text{ even}, \quad \lambda^i(d_r) = 0 \quad \text{for } i > 2.$$

Define $\lambda_t(\alpha) = \sum_{i \geq 0} \lambda^i(\alpha) t^i$. Then the Grothendieck γ -operations γ^i are obtained from the equality of the polynomials

$$\lambda_{t/(1-t)}(\alpha) = \gamma_t(\alpha) = \sum_{i \geq 0} \gamma^i(\alpha) t^i.$$

The following is well known (cf. [8, p. 2]).

LEMMA 3.1. $\gamma_t(\delta_{2r+1}) = 1 + \delta_{2r+1}(t - t^2)$, where $\delta_{2r+1} = d_{2r+1} - 2 \in \tilde{R}_C(Q_k)$.

Let v and z be the elements in $\tilde{R}_R(Q_k)$ defined in (1.2). Then we prove

LEMMA 3.2. $\gamma_t(v) = 1 + v(t - t^2) + z(t - t^2)^2$.

PROOF. Since γ_t is natural with respect to the complexification c_R , we have, by Lemma 3.1 and (1.2),

$$\begin{aligned} \gamma_t(v) &= \gamma_t c_R^{-1}(2\delta_1) = c_R^{-1}\{\gamma_t(\delta_1)\}^2 = c_R^{-1}\{1 + \delta_1(t - t^2)\}^2 \\ &= c_R^{-1}\{1 + 2\delta_1(t - t^2) + \delta_1^2(t - t^2)^2\} = 1 + v(t - t^2) + z(t - t^2)^2. \end{aligned}$$

As an application of Grothendieck γ -operations in KO -theory, M. F. Atiyah [1] obtained the following

THEOREM 3.3. Let M be an n -dimensional compact smooth manifold and $\tau_0 \in \widetilde{KO}(M)$ the stable class of the tangent bundle of M . Then, if M is immersible (resp. embeddable) in R^{n+r} , $\gamma^i(-\tau_0) = 0$ for all $i > r$ (resp. $i \geq r$).

LEMMA 3.4. Let τ_0 be the stable class of the tangent bundle $\tau = \tau(S^{4n+3}/Q_k)$ of S^{4n+3}/Q_k . Then

$$\gamma_t(-\tau_0) = \sum_{i \geq 0} \binom{2n+1+2i}{2i} z^i (t - t^2)^{2i} - \sum_{i \geq 0} \frac{1}{2} \binom{2n+2+2i}{2i+1} vz^i (t - t^2)^{2i+1}.$$

PROOF. According to [9, Corollary 3.3],

$$-\tau_0 = 4n + 3 - \tau = 4(n+1) - (n+1)(r_C \pi^* \lambda) = -(n+1)v.$$

By Lemma 3.2 and (1.2), we have

$$\begin{aligned}
 \gamma_t(-\tau_0) &= \gamma_t(-(n+1)v) = (\gamma_t(v))^{-n-1} = \{1 + v(t - t^2) + z(t - t^2)^2\}^{-n-1} \\
 &= c_R^{-1} \{1 + \delta_1(t - t^2)\}^{-2n-2} \\
 &= c_R^{-1} \left\{ \sum_{j \geq 0} (-1)^j \binom{2n+1+j}{j} \delta_1^j (t - t^2)^j \right\} \\
 &= c_R^{-1} \left\{ \sum_{i \geq 0} \binom{2n+1+2i}{2i} \delta_1^{2i} (t - t^2)^{2i} \right. \\
 &\quad \left. - \sum_{i \geq 0} \frac{1}{2} \binom{2n+2+2i}{2i+1} 2\delta_1^{2i+1} (t - t^2)^{2i+1} \right\} \\
 &= \sum_{i \geq 0} \binom{2n+1+2i}{2i} z^i (t - t^2)^{2i} - \sum_{i \geq 0} \frac{1}{2} \binom{2n+2+2i}{2i+1} v z^i (t - t^2)^{2i+1}.
 \end{aligned}$$

Now we are ready to prove Theorem 1.1.

PROOF OF THEOREM 1.1. Let n be even. Let $y_i = z^{i/2}$ if i is even and $y_i = -2^{-1}vz^{(i-1)/2}$ if i is odd. Then, by Theorem 1.6, $\#y_i = 2^{2n+k-2i}$ for $i \leq n$ and $y_i = 0$ for $i > n$. Also, by Lemma 3.4,

$$\begin{aligned}
 \gamma_t(-\tau_0) &= \sum_l \binom{2n+1+l}{l} y_l t^l (1-t)^l \\
 &= \sum_s (-1)^s t^s \sum_{2^{-1}s \leq l \leq s} (-1)^l \binom{l}{s-l} \binom{2n+1+l}{l} y_l.
 \end{aligned}$$

Hence, if S^{4n+3}/Q_k is immersed in $R^{4n+2+2i}$, then for all $s \geq 2i$,

$$\sum_{2^{-1}s \leq l \leq s} (-1)^l \binom{l}{s-l} \binom{2n+1+l}{l} y_l = 0.$$

The desired equalities

$$\binom{2n+1+l}{l} y_l = 0$$

are obtained by a downward induction on s , beginning with $s = 2n$.

The other cases are similar.

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